ი. ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი
ზუსტი და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი

ინტერდისციპლინური (მათემატიკა, კომპიუტერული მეცნიერებები)
ქვემიმართულება: მათემატიკური ლოგიკა და დისკრეტული სტრუქტურები

საფაკულტეტო კონფერენცია **2017**

რევაზ გრიგოლია , ანტონიო დი ნოლა ,
რამაზ ლიპარტელიანი

თავისუფალი და პროექციუ ლი ალგებრები სრულყოფილი მონადიკური MV-ალგებრების მრავალსახეობაში

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2017

Free and Projective algebras in the variety of monadic perfect MV-algebras

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ABSTRACT

A description and characterization of finitely generated free and projective monadic MV-algebras (MMV(C)-algebras) in the variety generated by monadic perfect MV -algebras is given. Finitely generated subdirectly irreducible MMV(C)-algebras are described.

აბსტრაქტი

დახასიათებულია და აღწერილია სასრულად წარმოქმნილი თავისუფალი და პროექციული მონადიკური MV-ალგებრები (MMV(C)-ალგებრები) სრულყოფილი მონადიკური MV-ალგებრებით წარმოქმნილ მრავალსახეობაში. აღწერილია სასრულად წარმოქმნილი ქვეპირდაპირად დაუშლადი MMV(C)-ალგებრები

Monadic MV-algebras

The infinitely valued propositional calculi, which have been described by Lukasiewicz and Tarski in 1930, are extended to the corresponding predicate calculi.

The predicate Lukasiewicz (infinitely valued) logic *QL* is defined in standard way.

Monadic MV-algebras

Monadic *MV*-algebras were introduced and studied by Rutledge in

[**J.D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus*, Ph.D. Thesis, Cornell University, 1959**.]

as an algebraic model for the predicate calculus QL of Lukasiewicz infinite valued logic, in which only a single individual variable occurs.

Monadic MV-algebras

Let L denote a first-order language based on $\cdot$, +, $\rightarrow$, $\neg$, $\exists$ and let L_m denote a propositional language based on $\cdot$, +, $\rightarrow$, $\neg$, $\exists$. Let $\mathrm{Form}(L)$ and $\mathrm{Form}(L_m)$ be the set of all formulas of L and L_m, respectively.

We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p*(x)$ in L and denote by induction a translation

$\Psi : \mathrm{Form}(L_m) \rightarrow \mathrm{Form}(L)$ by putting:

- $\Psi(p) = p*(x)$ if p is propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ \in \{\cdot, +, \rightarrow\}$,
- $\Psi(\exists \alpha) = \exists x \; \Psi(\alpha)$.

Monadic MV-algebras

An ***MV-algebra*** is an algebra

$$\mathbf{A} = (A, \oplus, \otimes, {}^*, 0, 1)$$

where $(A, \oplus, 0)$ is an *abelian monoid*, and for all $x, y \in A$ the following identities hold:

$x \oplus 1 = 1$, $x^{**} = x$,

$(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$,

$x \otimes y = (x^* \oplus y^*)^*$.

MV -algebras

It is well known that the *MV*-algebra $S = ([0, 1], \oplus, \otimes, *, 0, 1)$, where $x \oplus y = \min(1, x+y)$, $x \otimes y = \max(0, x+y-1)$, $x^* = 1-x$, generates the variety **MV** of all *MV*-algebras.

Let Q denote the set of rational numbers, for $(0 \neq)\ n \in \omega$ we set $S_n = (S_n, \oplus, \otimes, *, 0, 1)$, where $S_n = \{0, 1/n\text{-}1, \ldots, n\text{-}2/n\text{-}1, 1\}$ is also *MV*-algebra.

Monadic MV-algebras

An algebra A =(A, $\oplus$, $\otimes$, *, $\exists$, 0, 1) (also denoted as (A, $\exists$)) is said to be ***a monadic MV-algebra*** (for short MMV-algebra) [**A.Di Nola, R.Grigolia**] if (A, $\oplus$, $\otimes$, *, 0, 1) is an *MV*-algebra and in addition $\exists$ satisfies the following identities:

- E1. $x \leq \exists x$,
- E2. $\exists(x \vee y) = \exists x \vee \exists y$,
- E3. $\exists(\exists x)^* = (\exists x)^*$,
- E4. $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
- E5. $\exists(x \otimes x) = \exists x \otimes \exists x$,
- E6. $\exists(x \oplus x) = \exists x \oplus \exists x$.

Monadic MV -algebras

- A subalgebra A_0 of an *MV*-algebra *A* is said to be ***relatively complete*** if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has the least element, which is denoted by $inf\ \{b \in A_0\ a \leq b\}$.
- The *MV*-algebra $\exists A$ (= $\{\exists a : a \in A\}$) is a relatively complete subalgebra of the MV-algebra $(A, \oplus, \otimes, *, 0, 1)$, and $\exists a = inf\{b \in \exists A : a \leq b\}$.

Monadic MV -algebras

A subalgebra A_0 of an MV-algebra A is said to be ***m-relatively complete*** [**A.Di Nola, R.Grigolia**], if A_0 is relatively complete and two additional conditions hold:

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \otimes a \Rightarrow v \geq a \; \& \; v \otimes v \leq x)$,

(##) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq 0 \; \& \; v \oplus v \leq x)$.

Monadic MV-algebras

Let us remark that not every relatively complete subalgebra A_0 of an *MV*-algebra *A* defines a monadic operator ∃ converting *A* into a monadic MV-algebra.

Proposition 1. [**A.Di Nola, R.Grigolia**]. *The MV-algebra* $\exists A$ (= $\{\exists a$: $a \in A\}$) *is a m-relatively complete subalgebra of the MV-algebra* (A,⊕, ⊗, *, 0, 1). *Moreover,* (A,∃) *is a monadic MV-algebra iff* A_0 (=∃A) *is m-relatively complete subalgebra of A.*

Monadic MV -algebras

The characterization of monadic *MV*-algebras as a pair of *MV*-algebras, where one of them is a special kind of subalgebra, are given in

[**A. Di Nola, R. Grigolia, *On Monadic MV-alge-bras*, APAL, Vol. 128, Issues 1-3 (August 2004), pp. 125-139.**

L.P. Belluce, R. Grigolia and A. Lettieri, *Representations of monadic MV- algebras*", Studia Logica, vol. 81, Issue October 15th, 2005, pp.125-144.]

Monadic Boolean algebras

- Monadic Boolean algebras was introduced by Halmos in [**P.R. Halamos, *Algebraic Logic* (Chelsea, New York, 1962).**] as an algebraic counterpart of logical notion of a existential quantifier. Halmos [**P.R. Halamos, *Algebraic Logic I. Monadic Boolean algebras*, Compositio Math. 12 (1955), 217-249**] establish a duality between quantifiers on a Boolean algebra *A* and some equivalence relations on the Stone space of A

Q-distributive lattices

R. Cignoli [**R. Cignoli, *Quantiers on distributive lattices*, Discrete Mathematics, 96(1991), 183-197**.] have shown that there is duality between quantifiers on bounded distributive lattice *L* and certain equivalence relation on the Priestly space of *L*, and this duality reduces to that discovered by Halmos when restricted to Boolean algebras. Moreover, he was introduced bounded distributive lattices endowed with a quantifier as algebras, which named *Q-distributive lattices*.

MV-space

- A topological space *X* is said to be an *MV-space* if there exists an *MV*-algebra *A* such that Spec(*A*) (= the set of prime filters of the *MV* –algebra *A* equipped with spectral topology) and *X* are homeomorphic. It is well known that Spec(*A*) with the specialization order (which coincides with the inclusion between prime filters) forms a root system.

MV-space

Actually any *MV*-space is a Priestly space which is a root system. An *MV*-space is a Priestley space X such that $R(x)$ is a chain for any $x \in X$ and a morphism between *MV*-spaces is a strongly isotone map (or an *MV*-morphism), i. e. a continuous map $\varphi : X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

Free MMV(C)-algebras

Theorem 1. *Let* $A = \prod_{i \in I} A_i$ *be a direct product of the family of all subdirectly irreducible one-generated* MMV (C)-*algebras* A_i *with generators* $g_i \in A_i$ $(i \in I)$. *Let* $F_{\mathbf{MMV(C)}}(1)$ *be the subalgebra of A generated by* $g = (g_i)_{i \in I} \in A$. *Then*

1) *the algebra* $F_{\mathbf{MMV(C)}}(1)$ *is a subdirect product of the family* $\{A_i : i \in I\}$;
2) *any subdirectly irreducible one-generated MMV(C)-algebra is a homomorphic image of* $F_{\mathbf{MMV(C)}}(1)$;
3) *the algebra* $F_{\mathbf{MMV(C)}}(1)$ *generated by the generator* $g = (g_i)_{i \in I} \in A$ *is one-generated free MMV(C)-algebra with free generator* $g = (g_i)_{i \in I}$;
4) *the algebra* $F_{\mathbf{MMV(C)}}(1)$ *has height* 3;
5) *the poset of prime filters of the algebra* $F_{\mathbf{MMV(C)}}(1)$ *contains only four maximal elements and this four elements form the poset of MMV(C)-algebra* $(\mathbf{2}^2, \exists) \times (\mathbf{2}, \exists)^2$, where **2** is two-element Boolean algebra.

Proposition 2. [Bass1958, Grigolia1983,1987].

m-generated free monadic Boolean algebra ($B(m)$, ∃) *is isomorphic to*

$$\prod_{k=1}^{2^m} (2^k, \exists)^{\binom{k}{2^m}}$$

$\binom{k}{2^m}$

Corollary 3. *There exists exactly* $\sum_{k=1}^{2^m} \binom{k}{2^m} \, (= 2^{2^m} - 1)$ *number of maximal monadic filters of* $(B(m), \exists)$. *These maximal monadic filters are generated by* $(0^1, \ldots, 0^{k-1}, 1^k, 0^{k+1}, \ldots, 0^{2^m})$ *where* 1^k *is the top element of* $(\mathbf{2}^k, \exists)$ $(1 \le k \le 2^m)$, 0^i *is the bottom element of* $(\mathbf{2}^i, \exists)$ $(1 \le i \le 2^m)$.

Lemma 4. *The height of an m-generated sub-directly irreducible MMV(C)-algebra is limited by some natural number. In other words, a maximal chain of the poset of prime filters of a subdirectly irreducible MMV (C)-algebra is limited by some natural number* $k > 0$.

Lemma 4. *The height of an m-generated subdirectly irreducible MMV(C)-algebra is limited by some natural number. In other words, a maximal chain of the poset of prime filters of a subdirectly irreducible MMV (C)-al-gebra is limited by some natural number* $k > 0$.

Lemma 5. *There are infinitely many subdirectly irreducible m-generated MMV(C)-algebras for* $m > 1$.

Theorem 6. *The m-generated subdirectly irreducible MMV(C)-algebras for* $m \geq 2$ *are*:

1) $(2^{2^m}, \exists)$,
2) $(C_m, \exists)$,
3) $(C^{2^m}, \exists)$,
4) $(\mathrm{Rad}^*(C^m), \exists)$,
5) $(C_m^m, \exists)$.

.

Theorem 7. *Let* $A = \Pi_{i \in I} A_i$ *be a direct product of the family of subdirectly irreducible m-generated MMV(C)-algebras* A_i *with generators* $g_i^{(1)}, g_i^{(2)}, \dots, g_i^{(m)} \in A_i$ $(i \in I)$, *where* $\{g_i^{(1)}, g_i^{(2)}, \dots, g_i^{(m)}\} \neq \{g_j^{(1)}, g_j^{(2)}, \dots, g_j^{(m)}\}$ *for* $i \neq j$. Let $F_{\mathbf{MMV(C)}}(m)$ *be the subalgebra of A generated by the generators* $g_1 = (g_i^{(1)})_{i \in I} \in A, \dots\ g_m = (g_i^{(m)})_{i \in I} \in A$. *Then*

1) *the algebra* $F_{\mathbf{MMV(C)}}(m)$ *is a subdirect product of the family* $\{A_i : i \in I\}$;

2) *any subdirectly irreducible m-generated MMV(C)-algebra is a homomorphic image of* $F_{\mathbf{MMV(C)}}(m)$;

3) *the algebra* $F_{\mathbf{MMV(C)}}(m)$ *generated by the generator* $g_1 = (g_i^{(1)})_{i \in I} \in A, \dots\ g_m = (g_i^{(m)})_{i \in I} \in A$ *is m-generated free MMV(C)-algebra with free generator* $g_1 = (g_i^{(1)})_{i \in I} \in A, \dots\ g_m = (g_i^{(m)})_{i \in I} \in A$.

Projective MMV(C)-algebras

Theorem 8. *If A is n-generated projective MM V (C)-algebra, then A is finitely presented.*

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Lemma 10. The Boolean envelope $(B(m), \exists)$ of the algebra $F_{\mathbf{MMV(C)}}(m)$, where $B(m) = \{2x^2 : x \in F_{\mathbf{MMV(C)}}(m)\}$, is a retract of the algebra $F_{\mathbf{MMV(C)}}(m)$. In other words the m-generated monadic Boolean algebra $(B(m), \exists)$ is a projective algebra in **MMV(C)**.

Theorem 11. *m-generated subalgebra* $(A, \exists)$ *of* $F_{\mathbf{MMV(C)}}(m)$ *is projective iff* $(A,\exists)$ *is finitely presented and* A *is isomorphic to* $A_0 \times A_1$ *where* A_0 *is a perfect MV-algebra.*

THANK YOU