Free and Projective algebras in the variety of monadic perfect MV-algebras

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Abstract

A description and characterization of finitely generated free and projective monadic MV-alge-bras ($MMV(C)$-algebras) in the variety generated by monadic perfect MV -algebras is given. Finitely generated subdirectly irreducible $MMV(C)$-algebras are described.

Key Words and Phrases: MV-algebras, monadic MV-algebras, perfect MV-algebras.

1 Introduction

The finitely valued propositional calculi, which have been described by Łukasiewicz and Tarski in [16], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [20]. Scarpellini in [21] has proved that the set of valid formulas is not recursively enumerable.

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MV-algebras are the algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. In contrast to what happens for Boolean algebras, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of MV-algebra A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [4]. The logic of perfect MV-algebras that coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, see [4].

The importance of the variety generated by perfect MV-algebras and corresponding to it logic can be perceived by looking further at the role that infinitesimals play in MV-algebras and Łukasiewicz logic. Indeed the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0, 1]$, see [3, 21]. The Lindenbaum algebra of the first order LŁukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of the first order Łukasiewicz logic.

Monadic MV-algebras were introduced and studied by Rutledge in [20] as an algebraic model for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [20], showing the completeness of the monadic predicate calculus, has been of great interest.

Let L denote a first-order language based on $\cdot, +, \rightarrow, \neg, \exists$ and let L_m denote a propositional language based on $\cdot, +, \rightarrow, \neg, \exists$. Let $Form(L)$ and $Form(L_m)$ be the set of all formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \rightarrow Form(L)$ by putting:

- $\Psi(p) = p^*(x)$ if p is propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \rightarrow$,

- $\Psi(\exists\alpha) = \exists x\Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x.

For a detailed consideration of Łukasiewicz predicate calculus we refer to [1, 3, 12, 15, 16, 22, 23].

The paper is devoted to study of problems of projectivity and unification in he variety generated by monadic perfect MV-algebras. For this aim a description of finitely generated free algebras is given (Sections 3 and4). The characterization of finitely generated projective algebras in the variety generated by monadic perfect MV-algebras and and correspondence of projective algebras and projective formulas are given (Sections 5 and 6).

2 Preliminaries on Monadic MV-algebras

The characterization of monadic MV-algebras as pair of MV-algebras, where one of them is a special kind of subalgebra (m-relatively complete subalgebra), is given in [7, 5]. MV-algebras were introduced by Chang in [6] as an algebraic model for infinitely valued Łukasiewicz logic.

An MV-algebra is an algebra $A = (A, \oplus, \otimes, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \otimes y = (x^* \oplus y^*)^*$.

Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } x^* \oplus y = 1.$$

$(A, \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations: $x \oplus y = \min(1, x+y), x \otimes y = \max(0, x+y-1), x^* = 1-x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \otimes, ^*, 0, 1)$ generates the variety **MV** of all MV-algebras, i. e. $\mathcal{V}(S) = \mathbf{MV}$.

Let Q denote the set of rational numbers; then $[0, 1] \cap Q$ is another MV-algebra, which also generates the variety **MV**.

An algebra $A = (A, \oplus, \otimes, ^*, \exists, 0, 1)$ is said to be a monadic MV-algebra (MMV-algebra for short) [20, 7] if $A = (A, \oplus, \otimes, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities:

E1. $x \leq \exists x$,

E2. $\exists(x \vee y) = \exists x \vee \exists y$,

E3. $\exists(\exists x)^* = (\exists x)^*$,

E4. $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,

E5. $\exists(x \otimes x) = \exists x \otimes \exists x$,

E6. $\exists(x \oplus x) = \exists x \oplus \exists x$.

Sometimes we shall denote a monadic MV-algebra $A = (A, \oplus, \otimes, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. We can define a unary operation $\forall x = (\exists x^*)^*$ corresponding to the universal quantifier.

From the variety of monadic MV-algebras **MMV** [7] select the subvariety **MMV**(**C**) which is defined by the following equation [10]:

$$(Perf) \ \ 2(x^2) = (2x)^2,$$

that is $\mathbf{MMV}(\mathbf{C}) = \mathbf{MMV} + (Perf)$. The main object of our interest are the varieties **MMV**(**C**).

An ideal I (a filter F) of an algebra $(A, \exists) \in \mathbf{MMV}$ is called *monadic ideal (filter)* (see [R, DG]), if I (F) is an ideal (a filter) of MV-algebra A (i.e. $A \supset I \neq \emptyset$ $(A \supset F \neq \emptyset)$ and for every $x, y \in I$ $(x, y \in F)$ (a) $x \oplus y \in I$ $(x \otimes y \in F)$; (b) $x \geq y, x \in I \Rightarrow y \in I$ $(x \leq y, x \in F \Rightarrow y \in F)$) and for every $a \in A$ we have $a \in I \Rightarrow \exists a \in I$ $(a \in F \Rightarrow \forall a \in F)$. Notice that if I (F) is a monadic ideal (filter) of $(A, \exists)$, the the set $\{\neg x : x \in I\}$ $(\{\neg x : x \in F\})$ is a monadic filter (ideal).

For every monadic MV-algebra $(A, \exists)$, there exists a lattice isomorphism between the lattice of all monadic ideals (filters) and the lattice of all congruence relations of $(A, \exists)$ [7].

Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [4]. They generate the smallest non locally finite subvariety of the variety **MV** of all MV-algebras.

It is worth stressing that the variety generated by all perfect MV-algebras, denoted by **MV**(**C**), is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [6]. We name $MV(C)$*-algebras* all the algebras from the variety generated by C. Let L_P be the logic corresponding to the variety generated by perfect algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, or equivalently that are valid in the MV-algebra C. Actually, L_P is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom: $(x \veebar x)\&(x \veebar x) \leftrightarrow (x\&x) \veebar (x\&x)$ (where $\veebar$ is strong disjunction, & strong conjunction in Łukasiewicz sentential calculus), see [4]. Notice that C is a subalgebra of any non-Boolean perfect MV-algebra.

As it is well known, MV-algebras form a category that is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [18]. Let us denote by Γ the functor implementing this equivalence. If G is an ℓ-group, then for any element $u \in G$, $u > 0$ we let $[0, u] = \{x \in G : 0 \leq x \leq u\}$ and for each $x, y \in [0, u]$ $x \oplus y = u \wedge (x + y)$ and $\neg x = u - x$. In particular each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit.

Let us introduce some notations: let $C_0 = \Gamma(Z, 1)$, $C_1 = C \cong \Gamma(Z \times_{lex} Z, (1, 0))$ with generator $(0, 1) = c_1(= c)$, $C_m = \Gamma(Z \times_{lex} \cdots \times_{lex} Z, (1, 0, ..., 0))$ with generators $c_1(= (0, 0, ..., 1)), ..., c_m(= (0, 1, ..., 0))$, where the number of factors Z is equal to $m + 1$ and $\times_{lex}$ is the lexicographic product and Γ is well-known Mundici's functor translating a lattice ordered group with strong unit into MV-algebra. Let us denote $Rad(A) \cup \neg Rad(A)$ through $R^*(A)$, where $\neg Rad(A) = \{x^* : x \in Rad(A)$.

Let $(A, \oplus, \otimes, ^*, \exists, 0, 1)$ be a monadic MV-algebra. Let $\exists A = \{x \in A : x = \exists x\}$. $(\exists A, \oplus, \otimes, ^*, 0, 1)$ is an MV-subalgebra of the MV-algebra $(A, \oplus, \otimes, ^*, 0, 1)$, which is m-relatively complete subalgebra [7]. Any m-relatively complete subalgebra A_0 of the MV-algebra A defines a monadic operator $\exists$ on A: $\exists x = \bigwedge\{y \in A_0 : y \geq x\}$ [7]. Notice that two-element subalgebra of C (which is Boolean algebra)is not m-relatively complete. Indeed, for any element a $RadC$ we have $a \otimes a = 0$, $\exists a = 1$. But in this case $0 = \exists(a \otimes a) \neq \exists a \otimes \exists a = 1$, and it does not satisfies the axiom **E5**.

3 One-generated free monadic $MMV(C)$-algebras

According to the definition of monadic MV-algebras m-relatively complete subalgebra of C coincides with C but not its two-element Boolean subalgebra. In other words, $(C, \exists)$ is monadic $MMV(C)$-algebra if $\exists x = x$. Let we have

C^n for some non-negative integer. Then $(C^n, \exists)$ will be $MMV(C)$-algebra, where $\exists(a_1, ..., a_n) = max\{a_1, ..., a_n\}$ and $\forall(a_1, ..., a_n) = min\{a_1, ..., a_n\}$. In this case $\exists(C^n) = \{(x, ..., x) \in C^n : x \in C\}$. Notice, that $(C^n, \exists)$ is subdirectly irreducible [7]. For perfect MV-algebra $Rad^*(C^2)$ we also have $\exists(C^n) = \{(x, ..., x) \in C^n : x \in C\} \subset Rad^*(C^2)$.

Now we shall give examples of one-generated $MMV(C)$-algebras and show that there are infinitely many one generated subdirectly irreducible $MMV(C)$-algebras unlike the one generated subdirectly irreducible $MV(C)$-algebras which is only one (up to isomorphism) subdirectly irreducible $MV(C)$-algebra C.

Lemma 1. *The following algebras are one-generated subdirectly irreducible $MMV(C)$-algebras:*

1) $(\mathbf{2}, \exists)$ *with generator either* 1 *or* 0*, where* $\mathbf{2}$ *is two-element Boolean algebra ,*

2) $(\mathbf{2}^2, \exists)$ *with generator either* $(0,1)$ *or* $(1,0)$*, where* $\mathbf{2}^2$ *is four-element Boolean algebra,*

3) $(C, \exists)$ *with generator either* c *or* $\neg c$*,*

4) $(C^2, \exists)$ *with generator either* $(1, c), (\neg c, 0)$ *or* $(c, \neg c)$*,*

5) $(Rad^*(C^2), \exists)$ *with generator either* $(c, 0)$ *or* $(\neg c, 1)$*,*

6) $(C_2^2, \exists)$ *with generator either* $(c_1, \neg c_2)$ *or* $(\neg c_1, c_2)$*,*

7) $(Rad^*(C_2^2), \exists)$ *generated by* (c_1, c_2) *or* $(\neg c_1, \neg c_2)$*.*

Proof. 1), 2) and 3) is trivial.

4) (a) $\forall(1, c) = (c, c)$, $g^2 = (1, 0)$, $(c, c) \vee (0, 1) = (c, 1)$. So, $(C^2, \exists)$ is generated by $(1, c)$; (b) $2(\neg c, 0) = (1, 0)$, $\neg(\neg c, 0) = (c, 1)$, $(c, 1)^2 = (0, 1)$. So, $(C^2, \exists)$ is generated by $(\neg c, 0)$; (c) $2((c, \neg c)^2) = (0, 1)$, $\neg(0, 1) = (1, 0)$, $\forall(c, \neg c) = (c, c)$. So, $(C^2, \exists)$ is generated by $(c, \neg c)$;

5) $\exists(c, 0) = (c, c)$, $\neg(c, 0) = (\neg c, 1)$, $(c, c) \to (c, 0) = (1, \neg c)$, $\neg(1, \neg c) = (0, c)$. So, $(Rad^*(C^2), \exists)$ is generated by either $(c, 0)$ or $(\neg c, 1)$.

6) Let $g = (c_1, \neg c_2)$. $2g^2 = (0, 1)$, $\neg(2g^2) = (1, 0)$. $\forall g = (c_1.c_1$, $\neg\exists g = (c_2, c_2)$; $\forall g \wedge 2g^2 = (0, c_1)$; $\forall g \wedge \neg(2g^2) = (c_1, 0)$; $\neg\exists g \wedge 2g^2 = (0, c_2)$; $\neg\exists g \wedge \neg(2g^2) = (c_2, 0)$. In a similar way it is shown that $(C_2^2, \exists)$ is generated by $(\neg c_1, c_2)$ $(= \neg g)$.

7) Let $g = (c_1, c_2)$. From this element we obtain the following sequences of elements: $\forall g = (c_1, c_1)$, $\exists g = (c_2, c_2)$, $\neg\forall g = (\neg c_1, \neg c_1)$, $\neg\exists g = (\neg c_2, \neg c_2)$; $\neg\exists g \oplus g = (\neg c_2, \neg c_2) \oplus (c_1, c_2) = (\neg c_2 \oplus c_1, 1)$, $\neg g \oplus \forall g = (\neg c_1, \neg c_2) \oplus (c_1, c_1) = (1, \neg c_2 \oplus c_1)$; $(\neg\exists g \oplus g) \otimes \forall g = (\neg c_2 \oplus c_1, 1) \otimes (c_1, c_1) = (0, c_1)$,

$(\neg g \oplus \forall g) \otimes \forall g = (c_1, 0)$; $(\neg \exists g \oplus g) \otimes \neg \forall g) = (c_2, c_1)$, $\neg(\neg(c_2, c_1) \oplus (0, c_1)) = (c_2, 0)$, $\neg(\neg(c_1, c_2) \oplus (c_1, 0)) = (0, c_2)$. From these elements we can obtain all elements of radical of $(C_2^2, \exists)$ and thereby the elements of perfect algebra. $\square$

Notice that $(C_n, \exists)$ is not 1-generated for $n \geq 2$, since $\exists x = x$ for every $x \in C_n$ and C_n is not one-generated. It is clear that $(\mathbf{2}^n, \exists)$ is a homomorphic image of $(C^n, \exists)$. But $(\mathbf{2}^n, \exists)$ is not generated by one generator for $n \geq 3$. Indeed, for any element $x \in \mathbf{2}^n$ the operation $\exists$ is defined as follows: $\exists x = (1, ...1, 1) \in \mathbf{2}^n$ if $x \neq (0, 0, ..., 0) \in \mathbf{2^n}$ and $\exists x = (0, ...0, 0) \in \mathbf{2}^n$ in other case. So, $(\mathbf{2}^n, \exists)$ is one-generated if it is one-generated using only Boolean operations. But $\mathbf{2}^n$ is not generated by one generator if $n \geq 3$.

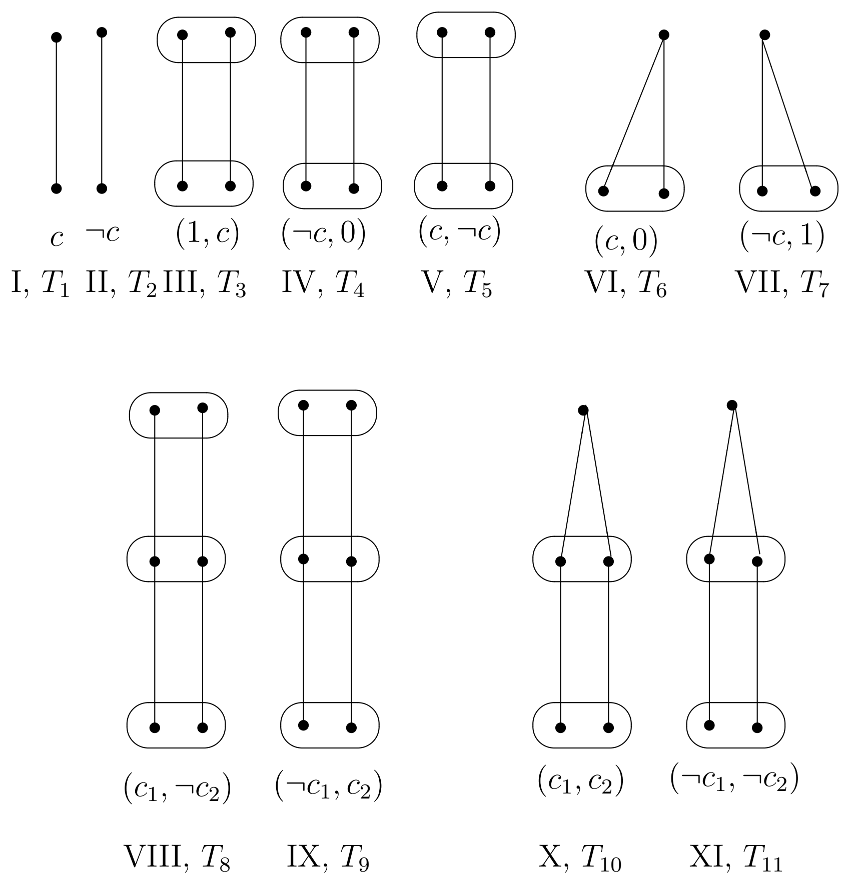


Fig. 1. Spectral spaces of one-generated subdirectly irreducible $MV(C)$-algebras

In Fig. 1 are depicted ordered sets corresponding to the prime filter spaces for $(C, \exists)$ ($\cong T_1 \cong T_2$), $(C^2, \exists)$ ($\cong T_3 \cong T_4 \cong T_5$), $(Rad^*(C^2), \exists)$ ($\cong T_6 \cong T_7$), $(C_2^2, \exists)$ ($\cong T_8 \cong T_9$), $(Rad^*(C_2^2), \exists)$ ($\cong T_{10} \cong T_{11}$) with their generators. Notice that the algebras $T_1, T_2, ..., T_7$ have height 2 and the algebras T_8, T_9, T_{10}, T_{11} have height 3 (the definition of height see below).

Let us give some comments about the diagrams in Fig. 1. The posets I, II, VI, VII, X and XI have one maximal filter, i. e. they correspond to a perfect MV -algebras. As to III, IV, V, VII and IX the elements being inside ovals we can consider as equivalent elements and this equivalence relation corresponds to the $\exists$ operation on a corresponding MV -algebra which corresponds to the diagonal subalgebra of C^2 and C_2^2 respectively.

We say that MV-algebra A has *height* n if a maximal chain of the poset of prime filters (ordered by inclusion) contains n elements. Similarly, we say that $MMV(C)$-algebra A has *height* n if its MV-algebra reduct has height n. According to this definition MV-algebra C_n has height $n+1$ ($n \geq 1$).

Lemma 2. *If subdirectly irreducible $MMV(C)$-algebra, with non-trivial operation $\exists$, has height $n > 3$, then it is not one-generated.*

Proof. Let us suppose we have $MMV(C)$-algebra $(C_3^2, \exists)$. The optimal version to be a generator of $(C_3^2, \exists)$ is either (c_1, c_2), (c_2, c_3), (c_1, c_3), $(c_1, \neg c_2)$, $(c_1, \neg c_3)$, $(\neg c_1, c_2)$, $(\neg c_1, c_3)$, $(\neg c_2, c_3)$, $(c_2, \neg c_3)$, $(c_2, \neg c_1)$, $(c_3, \neg c_2)$. It is obvious that none of them generates the algebra $(C_3^2, \exists)$. Even more so $(C_n^k, \exists)$ is not generated by one generator for $k, n > 2$.

□

The next lemma shows that there are infinitely many non-isomorphic one-generated subdirectly irreducible $MMV(C)$-algebras.

Lemma 3. *The $MMV(C)$-algebra $(Rad^*(C^n), \exists)$ is generated by the element $(c, 2c, ..., nc)$ for any positive integer n.*

Proof. Let $g = (c, 2c, ..., nc)$. Then $\forall g = (c, c, ..., c)$, $\neg\forall g = (\neg c, \neg c, ..., \neg c)$, $g \otimes \neg\forall g = (0, c, 2c, ..., (n-1)c)$, $g \otimes (\neg\forall g)^2 = (0, 0, c, 2c, ..., (n-2)c)$, ... , $g \otimes (\neg\forall g)^{n-1} = (0, 0, ..., 0, c)$.

$(g \otimes \neg\forall g) \wedge \forall g = (0, c, ..., c)$, $\neg(g \otimes (\neg\forall g)^2) \otimes ((g \otimes \neg\forall g) \wedge \forall g) = (1, 1, \neg c, ..., (\neg c)^{n-2}) \otimes (0, c, ..., c) = (0, c, 0, ..., 0)$.

$(g \otimes (\neg\forall g)^2) \wedge \forall g = (0, 0, c, ..., c)$. $\neg(g \otimes (\neg\forall g)^3) = (1, 1, 1, \neg c, (\neg c)^2, ..., (\neg c)^{n-3})$. $(1, 1, 1, \neg c, (\neg c)^2, ..., (\neg c)^{n-3}) \otimes (0, 0, c, ..., c) = (0, 0, c, 0, ..., 0)$, and so on.

Moreover, $\neg g \otimes 2\forall g = (c, 0, ..., 0)$. From here we conclude the proof of the theorem.

□

Lemma 4. *$MMV(C)$-algebra*

$$U_1 = Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))(= Rad^*(T_7 \times T_1))$$

is generated by $((c, 0), c)$ $(((\neg c, 1), \neg c))$, which is a perfect MV-algebra. Moreover, the subalgebra of $Rad^*(C^2, \exists) \times (C, \exists))$ generated by $((c, 0), c)$ $(((\neg c, 1), \neg c))$ is isomorphic to $Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$.

Proof. It is clear that by means of elements $((0, 0), c)$, $((0, c), 0)$, $((c, 0), 0)$ and the operations $\oplus$, $\vee$ we can obtain all elements of $Rad((Rad^*(C^2, \exists) \times (C, \exists))$, and thereby all elements of $Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$. Let $g = ((c, 0), c)$. Then $\forall g = ((0, 0), c)$; $\exists g = ((c, c), c)$; $\neg\forall g = ((1, 1), \neg c)$; $\neg g = ((\neg c, 1), \neg c)$; $\neg\forall g \otimes \exists g = ((c, c), 0)$; $(\neg\forall g \otimes \exists g) \to g = ((1, \neg c), 1)$; $\neg\forall g \otimes \exists g = ((c, c), 0)$; $\neg((\neg\forall g \otimes \exists g) \to g) = ((0, c), 0)$; $((c, 0)c) \wedge ((c, c), 0) = ((c, 0), 0)$. So we have obtained the elements $((0, 0), c)$, $((0, c), 0)$, $((c, 0), 0)$. Hence, $Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$ is generated by $((c, 0), c)$, and thereby by element $(\neg c, 1), \neg c)$.

Observe that the element $((c, 0), c)$ $((\neg c, 1), \neg c))$ belongs to radical (coradical). So, the subalgebra generated by this element is perfect and isomorphic to $Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$.

□

Lemma 5. *$MMV(C)$-algebra*

$$U_1^2 = Rad^*((Rad^*(C^2, \exists) \times (C, \exists))) \times Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$$

is generated by $((c, 0), c), (\neg c, 1), \neg c))$.

Proof. Indeed, from the generator $((c, 0), c), (\neg c, 1), \neg c))$ we can obtain the elements $((0, 0), 0), (1, 1), 1)) = 2(((c, 0), c), (\neg c, 1), \neg c))^2)$ and $((1, 1), 1), (0, 0), 0)) = \neg((0, 0), 0), (1, 1), 1))$. So, $Rad^*((Rad^*(C^2, \exists) \times (C, \exists))) \times Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$ is generated by $((c, 0), c), (\neg c, 1), \neg c))$. □

Lemma 6. *The subalgebra U_2 of $MMV(C)$-algebra $(C^2, \exists)^3$ generated by $t = ((c, \neg c), (1, c), (\neg c, 0))$ is a proper subalgebra with one maximal monadic filter.*

Proof. Let us notice that one-generated non-trivial monadic Boolean algebra is isomorphic to $(2^2, \exists)$ with generator $(0, 1)$. Notice also that $(2^2, \exists) \cong (C^2, \exists)/((c, c)]$, where $((c, c)]$ is the monadic ideal generated by (c, c) which is maximal at the same time. So, since $(2^2, \exists)$ should be homomorphic image of the subalgebra of $MMV(C)$-algebra $(C^2, \exists)^3$ generated by $((c, \neg c), ((c, 0), c), ((\neg c, 1), \neg c))$, the subalgebra must have one maximal monadic ideal. Moreover, U_2 is a subdirect product of subdirectly irreducible copies of algebra $(C^2, \exists)$, since $(C^2, \exists)$ is generated separately by $(c, \neg c), (1, c), (\neg c, 0)$.

□

Lemma 7. *$U_2/J_i \cong (C^2, \exists)$ $(i = 1, 2, 3)$, where $J_1 = (((c, c), (0, 0), (0, 0))]$, $J_2 = (((0, 0), (c, c), (0, 0))]$, $J_3 = (((0, 0), (0, 0), (c, c))]$, that is the monadic ideals generated by $((c, c), (0, 0), (0, 0)), ((0, 0), (c, c), (0, 0)), ((0, 0), (0, 0), (c, c))$, respectively.*

Proof. Now we show that the elements can be obtained by the generator t. Indeed, $\neg\exists t \wedge \forall t = ((c, c), (0, 0), (0, 0))$; $(\neg\exists t \vee \forall t) \wedge \exists\neg t = ((0, 0), (0, 0), (c, c))$; $(\neg\exists t \oplus (\neg\exists t \vee \forall t)) \otimes (\neg(\exists\neg t \otimes (\neg\exists t \wedge \forall t) \wedge (\neg\exists t \wedge \forall t)^2 = ((0, 0), (c, c), (0, 0))$. □

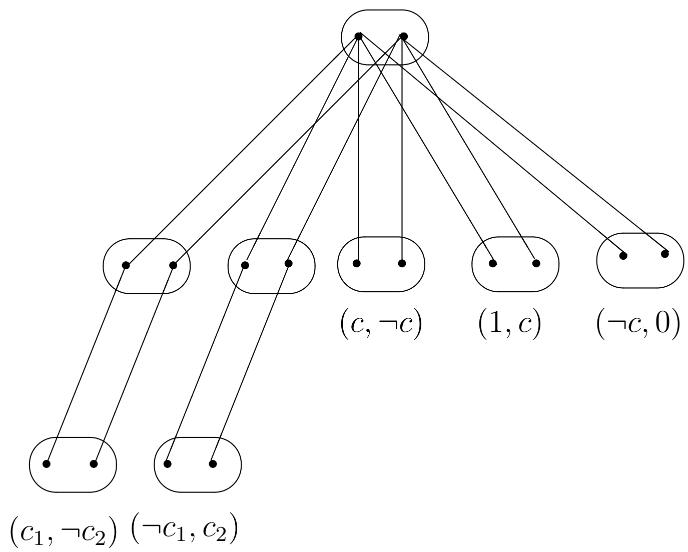


Fig. 2

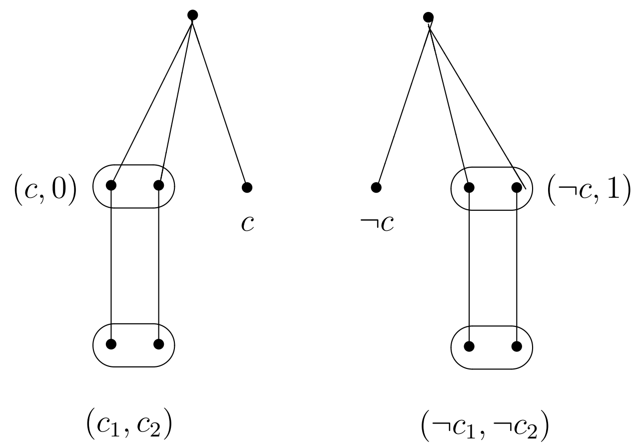


Fig. 3

The ordered set corresponding to the prime filter space of algebras $T_8 \times T_9 \times T_3 \times T_4 \times T_5$ generated by $(c_1, \neg c_2), (\neg c_1, c_2), (c, \neg c), (1, c), (\neg c, 0)$ is depicted in Fig. 2 and the ordered set corresponding to the prime filter space of algebras generated by $(c_1, c_2), c, \neg c, (\neg c_1, \neg c_2)$ is depicted in Fig. 3.

Theorem 8. *Let $A = \prod_{i \in I} A_i$ be a direct product of the family of all subdirectly irreducible one-generated $MMV(C)$-algebras A_i with generators $g_i \in A_i$ $(i \in I)$. Let $F_{\boldsymbol{MMV(C)}}(1)$ be the subalgebra of A generated by the generator $g = (g_i)_{i \in I} \in A$. Then*

1) the algebra $F_{\boldsymbol{MMV(C)}}(1)$ is a subdirect product of the family $\{A_i : i \in I\}$;

2) any subdirectly irreducible one-generated $MMV(C)$-algebra is a homomorphic image of $F_{\boldsymbol{MMV(C)}}(1)$;

3) the algebra $F_{\boldsymbol{MMV(C)}}(1)$ generated by the generator $g = (g_i)_{i \in I} \in A$ is one-generated free $MMV(C)$-algebra with free generator $g = (g_i)_{i \in I}$;

4) the algebra $F_{\boldsymbol{MMV(C)}}(1)$ has height 3;

5) the poset of prime filters of the algebra $F_{\boldsymbol{MMV(C)}}(1)$ contains only four maximal elements and this four elements form the poset of $MMV(C)$-algebra $(\mathbf{2}^2, \exists) \times (\mathbf{2}, \exists)^2$, where $\mathbf{2}$ is two-element Boolean algebra.

Proof. 1). It is obvious that for any projection π_i $(i \in I)$ $\pi_i(g) = g_i$ that generates A_i. So, $F_{\mathbf{MMV(C)}}(1)$ is a subdirect product of the family $\{A_i : i \in I\}$.

2). Because $F_{\mathbf{MMV(C)}}(1)$ is a subdirect product of all subdirectly irreducible one-generated $MMV(C)$-algebras A_i we have that any subdirectly irreducible one-generated $MMV(C)$-algebra is a homomorphic image of $F_{\mathbf{MMV(C)}}(1)$

3). Let us suppose that an identity $P(x) = Q(x)$ does not hold in the variety **MMV(C)**. Then it does not hold in some subdirectly irreducible one-generated $MMV(C)$-algebras A_i on the generator g_i. So, it does not hold in $F_{\mathbf{MMV(C)}}(1)$ on the generator g. From here we conclude that $F_{\mathbf{MMV(C)}}(1)$ generated by the generator $g = (g_i)_{i \in I} \in A$ is one-generated free $MMV(C)$-algebra with free generator $g = (g_i)_{i \in I}$.

4). The assertion follows from the Lemma 2.

5). This item follows from the fact that the algebra $(\mathbf{2}^2, \exists) \times (\mathbf{2}, \exists)^2$ is a free one-generated monadic Boolean algebra and the variety of monadic Boolean algebras is a subvariety of the variety **MMV(C)**.

□

4 m-generated free monadic $MMV(C)$-algebras

We can generalize easily the results of one-generated $MMV(C)$-algebras on m-generated. Since the prime filter space of 1-generated free $MMV(C)$-algebra and, also, m-generated free $MV(C)$-algebra ($m > 1$) is infinite [9], we have that the prime filter space of m-generated free $MMV(C)$-algebra is also infinite. But the number of the prime filter spaces of m-generated subdirectly irreducible $MMV(C)$-algebra is finite.

Notice that the smallest subvariety of the variety **MMV(C)**, different from the variety of Boolean algebras with trivial monadic operator, is the variety of monadic Boolean algebras. So, any m-generated free monadic Boolean algebra is a homomorphic image of m-generated free $MMV(C)$-algebra. It holds the following

Proposition 9. *[2, 13, 14]. m-generated free monadic Boolean algebra $(B(m), \exists)$ is isomorphic to*

$$\prod_{k=1}^{2^m} (\mathbf{2}^k, \exists)^{\binom{k}{2^m}}.$$

Corollary 10. *There exists exactly $\sum_{k=1}^{2^m} \binom{k}{2^m} (= 2^{2^m} - 1)$ number of maximal monadic filters of $(B(m), \exists)$. These maximal monadic filters are generated* *by* $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$ *where* 1^k *is the top element of* $(\mathbf{2}^k, \exists)$ *(*$1 \leq k \leq 2^m$*),* 0^i *is the bottom element of* $(\mathbf{2}^i, \exists)$ *(*$1 \leq i \leq 2^m$*).*

Notice, that monadic Boolean algebras are also monadic $MV(C)$-algebra, but of height 1.

As for one-generated case as an obvious fact we have the following

Lemma 11. *The height of an* m*-generated subdirectly irreducible* $MMV(C)$*-algebra is limited by some natural number* $k > 0$*. In other words, a maximal chain of the poset of prime filters of a subdirectly irreducible* $MMV(C)$*-algebra is limited by some natural number* $k > 0$*.*

Since we have infinitely many subdirectly irreducible one-generated $MMV(C)$-algebras, it holds

Lemma 12. *There are infinitely many subdirectly irreducible* m*-generated* $MMV(C)$*-algebras for* $m > 1$*.*

Theorem 13. *The* m*-generated subdirectly irreducible* $MMV(C)$*-algebras for* $m \geq 2$ *are:*

1) $(\mathbf{2}^{2^m}, \exists)$,

2) $(C_m, \exists)$,

3) $(C^{2^m}, \exists)$,

4) $(Rad^*(C^m), \exists)$,

5) $(C_m^m, \exists)$.

Proof. 1) and 2) is trivial. 3). It is obvious that $(C^{2^m}, \exists)$ has as a subalgebra the monadic Boolean algebra $(\mathbf{2}^{2^m}, \exists)$ the generators of which are the generators of the free m-generated Boolean algebra $\mathbf{2}^{2^m}$. If we change in every free generator of $\mathbf{2}^{2^m}$ the element 0 by c and 1 by $\neg c$, then we will get m generators of $(C^{2^m}, \exists)$. 4). It is obvious that $(c, 0, ..., 0), (0, c, , ..., 0), ..., (0, ..., c)$ generate $(Rad^*(C^m), \exists)$. 5). The generators of $(C_m^m, \exists)$ are $g_1 = (\neg c_1, c_2, ..., c_m$, $g_2 = (c_1, \neg c_2, ..., c_m$, ... , $g_m = (c_1, c_2, ..., \neg c_m$. Indeed, $\neg \exists g_1 = (c_1, c_1, ..., c_1)$, $\neg \exists g_2 = (c_2, c_2, ..., c_2)$, ... , $\neg \exists g_m = (c_m, c_m, ..., c_m)$; $2g_1^2 = (1, 0, ..., 0)$, $2g_2^2 = (0, 1, ..., 0)$, ... , $2g_m^2 = (0, 0, ..., 1)$. And these elements generate $(C_m^m, \exists)$.

□

Theorem 14. *Let* $A = \prod_{i \in I} A_i$ *be a direct product of the family of all subdirectly irreducible* m*-generated* $MMV(C)$*-algebras* A_i *with generators*

$g_i^{(1)}, g_i^{(2)}, ..., g_i^{(m)} \in A_i$ $(i \in I)$, where $\{g_i^{(1)}, g_i^{(2)}, ..., g_i^{(m)}\} \neq \{g_j^{(1)}, g_j^{(2)}, ..., g_j^{(m)}\}$ for $i \neq j$. Let $F_{\boldsymbol{MMV(C)}}(m)$ be the subalgebra of A generated by the generators $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$. Then

1) the algebra $F_{\boldsymbol{MMV(C)}}(m)$ is a subdirect product of the family $\{A_i : i \in I\}$;

2) any subdirectly irreducible m-generated $MMV(C)$-algebra is a homomorphic image of $F_{\boldsymbol{MMV(C)}}(m)$;

3) the algebra $F_{\boldsymbol{MMV(C)}}(m)$ generated by the generator $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$ is m-generated free $MMV(C)$-algebra with free generator $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$.

Proof. The theorem is proved as in one-generated case.

□

Theorem 15. *Free algebra $F_{\boldsymbol{MMV(C)}}(m)$ is isomorphic to the finite product of monadic $MV(C)$-algebras D_k $(1 \leq k \leq 2^{2^m} - 1)$ the homomorphic image by maximal monadic filter of which is isomorphic to the subdirectly irreducible monadic Boolean algebra $(\mathbf{2}^{m(k)}, \exists)$, where $m(k) \leq 2^m$. The number of subdirectly irreducible $MMVC)$-algebras having maximal homomorphic image the algebra $\mathbf{2}^{m(k)}$ is equal to $\binom{m(k)}{2^m}$.*

Proof. Notice that m-generated monadic Boolean algebra $(B(m), \exists)$ is a homomorphic image of $F_{\mathbf{MMV(C)}}(m)$. The algebra $(B(m), \exists)$ contains $2^{2^m} - 1$ maximal monadic filters. The intersection of all maximal monadic filters of $(B(m), \exists)$ is equal to $[1_{B(m)})$. According to Corollary 10 these maximal monadic filters of $(B(m), \exists)$ is generated by $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$ where 1^k is the top element of $(\mathbf{2}^k, \exists)$ $(1 \leq k \leq 2^m)$, 0^i is the bottom element of $(\mathbf{2}^i, \exists)$ $(1 \leq i \leq 2^m)$. Denote the maximal monadic filters of $(B(m), \exists)$ generated by $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$ by F_k. The factor algebra $(B(m)/F_k, \exists)$ is isomorphic to $(\mathbf{2}^k, \exists)$ that is subdirectly irreducible the number of which is equal to $\binom{k}{2^m}$. Let F_k^M be the monadic filter of $F_{\mathbf{MMV(C)}}(m)$ generated in $F_{\mathbf{MMV(C)}}(m)$ by F_k. It is obvious that the intersection of all such kind of the monadic filters of $F_{\mathbf{MMV(C)}}(m)$ is also equal to the unite element of $F_{\mathbf{MMV(C)}}(m)$. So, $F_{\mathbf{MMV(C)}}(m)$ is isomorphic to the finite product of the algebras $D_k = F_{\mathbf{MMV(C)}}(m)/F_k^M$, where $1 \leq k \leq 2^{2^m} - 1$.

□

5 Finitely generated projective $MMV(C)$-algebras

In this section previously we will prove auxiliary assertions.

Let $\mathbf{V}$ be a variety. Recall that an algebra $A \in \mathbf{V}$ is said to be a *free algebra* over $\mathbf{V}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{V}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite, then A is said to be a *free algebra of finitely many generators*. We denote a free algebra A with $m \in (\omega + 1)$ free generators by $F_{\mathbf{V}}(m)$. We shall omit the subscript $\mathbf{V}$ if the variety $\mathbf{V}$ is known.

An algebra A is called *projective* if for any algebra epimorfism (=homomorphism onto) $f : D \to B$ and a homomorphism $h : A \to B$ there is a homomorphism $g : A \to D$ such that $fg = h$. An algebra H is a retract of an algebra A if there are homomorphisms $f : A \to H$ and $g : H \to A$ such that $fg = Id_H$, where Id_H is an identity mapping of the set H. It is well-known that in varieties the projective algebras are just the retracts of the free algebras. So, a $MMV(C)$-algebra is projective if and only if it is a retract of a free $MMV(C)$-algebra. We say that the subalgebra A of $F_{\mathbf{V}}(m)$ is *projective* if there exists endomorphism $h : F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(m)$ such that $h(x) = x$ for every $x \in A$.

An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, ..., a_m \in A$, and there exist a finite number of equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ holding in A on the generators $a_1, ..., a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, ..., b_m \in B$, such that the equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ hold in B on the generators $b_1, ..., b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

Proposition 16. *[17] [7]. An m-generated algebra A in a variety $\mathbf{V}$ is projective if, and only if, there exist polynomials $P_1, \ldots, P_m$ such that, denoting by $g_1, ..., g_m$ the free generators of $F_{\mathbf{V}}(m)$,*

$P_i(P_1(g_1, \ldots, g_m), \ldots, P_m(g_1, \ldots, g_m)) = P_i(g_1, \ldots, g_m)$, *for each* $1 \leq i \leq m$ *and*

$P_1(g_1, \ldots, g_m), \ldots, P_m(g_1, \ldots, g_m)$ *generate an algebra isomorphic to A.*

Theorem 17. *If A is n-generated projective $MMV(C)$-algebra, then A is finitely presented.*

Proof. Since A is n-generated projective $MMV(C)$-algebra, A is retract of $F_{\mathbf{MMV(C)}}(n)$, i. e. there exist homomorphisms $h : F_{\mathbf{MMV(C)}}(n) \to A$ and $\varepsilon : A \to F_{\mathbf{MMV(C)}}(n)$ such that $h\varepsilon = Id_A$, and moreover, there exist n polynomials $P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)$ such that

$$P_i(g_1, \ldots, g_n) = \varepsilon(a_i) = \varepsilon h(g_i)$$

and

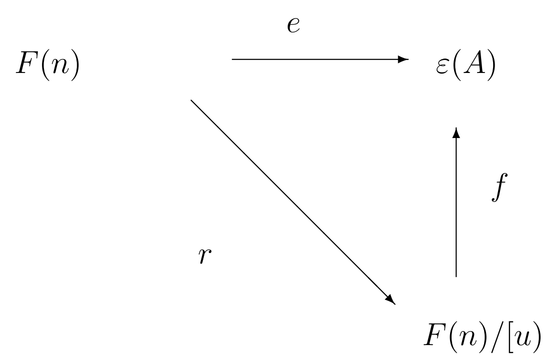
$$P_i(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)) = P_i(x_1, \ldots, x_n),\ i = 1, \ldots, n,$$

where $g_1, \ldots, g_n$ are free generators of $F_{\mathbf{MMV(C)}}(n)$. Observe that $h(g_1), \ldots, h(g_n)$ are generators of A which we denote by $a_1, \ldots, a_n$ respectively. Let e be the endomorphism $\varepsilon h : F_{\mathbf{MMV(C)}}(n) \to F_{\mathbf{MMV(C)}}(n)$. This endomorphism has properties : $ee = e$ and $e(x) = x$ for every $x \in \varepsilon(A)$.

Let us consider the set of equations $\Omega = \{P_i(x_1, \ldots, x_n) \leftrightarrow x_i = 1 : i = 1, \ldots, n\}$ and let $u = \bigwedge_{i=1}^{n}((P_i(g_1, \ldots, g_n) \leftrightarrow g_i) \in F(n)$, where $x \leftrightarrow y$ is abbreviation of $(x \to y) \wedge (y \to x)$. Observe that the equations from Ω are true in A on the elements $\varepsilon(a_i) = e(g_i),\ i = 1, \ldots, n$. Indeed, since e is an endomorphism

$$e(u) = \bigwedge_{i=1}^{n} e(g_i) \leftrightarrow P_i(e(g_1), \ldots, e(g_n)).$$

But $P_i(e(g_1), \ldots, e(g_n)) = P_i(P_1(g_1, \ldots, g_n), \ldots, P_n(g_1, \ldots, g_n)) = P_i(g_1, \ldots, g_n) = \varepsilon h(g_i) = e(g_i), i = 1, \ldots, n$. Hence $e(u) = 1$ and $u \in e^{-1}(1)$, i. e. $[u) \subseteq e^{-1}(1)$. Therefore there exists homomorphism $f : F(n)/[u) \to \varepsilon(A)$ such that the diagram



commutes, i. e. $rf = e$, where r is a natural homomorphism sending x to $x/[u]$. Now consider the restrictions e' and r' on $\varepsilon(A) \subseteq F(n)$ of e and r respectively Then $fr' = e'$. But $e' = Id_{\varepsilon(A)}$. Therefore $fr' = Id_{\varepsilon(A)}$. From here we conclude that r' is an injection. Moreover r' is a surjection, since $r(\varepsilon(a_i)) = r(g_i)$. Indeed $e(g_i) = P_i(g_1, \ldots, g_n)$ and $g_i \leftrightarrow P_i(g_1, \ldots, g_n) = g_i \leftrightarrow e(g_i)$, where $e(g_i) = \varepsilon h(g_i)$. So $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \geq \bigwedge_{i=1}^n g_i \leftrightarrow P_i(g_1, \ldots, g_n)$, i. e. $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \in [u)$. Hence r' is an isomorphism between $\varepsilon(A)$ and $F(n)/[u)$. Consequently $A(\cong \varepsilon(A))$ is finitely presented. $\square$

It easy to prove the following

Lemma 18. *Any m-generated non-Boolean subdirectly irreducible $MMV(C)$-algebra A contains $(C, \exists)$ as a subalgebra.*

Lemma 19. *Any subdirectly irreducible m-generated $MMV(C)$-algebra $(A, \exists)$ is a subalgebra of $(C_n^k, \exists)$ for some $n, k \in \omega$ and $n \leq m$.*

Proof. Let $(A, \exists)$ be subdirectly irreducible m-generated $MMV(C)$-algebra. Since $(A, \exists)$ is subdirectly irreducible, we have that $\exists A$ is totally ordered which is isomorphic to $(C_n, \exists)$ for some $n \leq m$. Then A as $MV(C)$-algebra is subdirect product of copies of C_n, i .e. A is a subalgebra of C_n^k for some $n, k \in \omega$ and $n \leq m$. Therefore, $(A, \exists)$ is a subalgebra of $(C_n^k, \exists)$, where the operation $\exists$ in $(A, \exists)$ is defined in the same way as in $(C_n^k, \exists)$. $\square$

Lemma 20. *The algebra $(C_m^k, \exists)$ is a retract of $(C_n^k, \exists)$ for any positive integer k, $1 \leq m \leq n$.*

Proof. Notice that $(C_m, \exists)$ is a subalgebra of $(C_n, \exists)$. So, we can define the embedding $\varepsilon : C_m^k \to C_n^k$ in the following way: $\varepsilon(a_1, ..., a_k) = (\varepsilon(a_1), ..., \varepsilon(a_k))$, where $\varepsilon(c_i) = c_{n-m+i}$ for $i = 1, ..., m$.

Let $h : C_n^k \to C_m^k$ be the homomorphism corresponding to the principal ideal generated by $(c_{n-m}, ..., c_{n-m})$. By this homomorphism we have $h(0) = h(c_i) = 0$ for $i = 1, ..., n-m$ and $h(c_{n-m+1}) = c_1,\ h(c_{n-m+2}) = c_2, ..., h(c_n) = c_m$. Then it is easy to check that $h\varepsilon = Id_{C_m^k}$, i. e. $(C_m^k, \exists)$ is a retract of $(C_n^k, \exists)$.

□

Lemma 21. *Let $(A, \exists)$ be m-generated subdirectly irreducible $MMV(C)$-algebra and $(u] \subset A$ principal monadic ideal generated by $u \in A$. Then $(A, \exists)/(u]$ is a retract of $(A, \exists)$.*

Proof. The algebra $(A, \exists)$ is a subalgebra of $(C_n^k, \exists)$ for some $n, k \in \omega$ and $n \leq m$ (Lemma 17) and as an MV-algebra A is a subdirect product of copies of C_n, $n \leq m$. Then for some $m \leq n$, we have $u = (c_{m-n}, ..., c_{m-n}) \in C_n^k$, since $(c_{m-n}, ..., c_{m-n}) \in \exists A$. Let h be the homomorphism corresponding to the principal ideal $(u]$. So, we have a homomorphism $h : C_n^k \to C_m^k$ such that $h(0) = h(c_i) = 0$ for $i = 1, ..., m-n$ and $h(c_{m-n+1}) = c_1,\ h(c_{m-n+2}) = c_2, ..., h(c_m) = c_n$.

Define the embedding $\varepsilon : C_n^k \to C_m^k$ in the following way: $\varepsilon(a_1, ..., a_k) = (\varepsilon(a_1), ..., \varepsilon(a_k))$, where $\varepsilon(c_i) = c_{m-n+i}$ for $i = 1, ..., m$. Then it is easy to check that $h\varepsilon = Id_A/9(u]$, i. e. $(A, \exists)/(u]$ is a retract of $(A, \exists)$.

□

Lemma 22. *Let $A \subset \prod_{i \in I} A_i$ be m-generated $MMV(C)$-algebra which is subdirect product of the family $\{A_i\}_{\in I}$ of the subdirectly irreducible algebras A_i $(i \in I)$ and $A'_i \subset A$, which is a retract of A_i for $i \in I$. Then subalgebra $A' = A \cap \prod_{i \in I} A'_i$ is a retract of A.*

Proof. Since A'_i is a retract of A_i, we have that there exist homomorphisms $\varepsilon_i : A'_i \to A_i$ and $h_i : A_i \to A'_i$ such that $h_i\varepsilon_i = Id_{A'_i}$. It is obvious that $\prod_{i \in I} A'_i$ is a retract of $\prod_{i \in I} A_i$. Indeed, there exist homomorphisms $h = (h_i)_{i \in I} : \prod_{i \in I} A_i \to \prod_{i \in I} A'_i$ and $\varepsilon = (\varepsilon_i)_{i \in I} : \prod_{i \in I} A'_i \to \prod_{i \in I} A_i$ such that $h\varepsilon = Id_{\prod_{i \in I} A'_i}$. Then the restriction of the homomorphism h on A, denoted by h_A, and the restriction of the homomorphism ε on A', denoted by ε_A, give $h_A\varepsilon_{A'} = Id_{A'}$.

□

Proposition 23. *[19]. m-generated monadic Boolean algebra* $(B, \exists)$ *is projective in the variety of monadic Boolean algebras iff* $(B, \exists) \cong (\mathbf{2}, \exists) \times (B', \exists)$ *for some m-generated monadic Boolean algebra* $(B', \exists)$.

Lemma 24. *The Boolean envelope* $(B(m), \exists)$ *of the algebra* $F_{\mathbf{MMV(C)}}(m)$, *where* $B(m) = \{2x^2 : x \in F_{\mathbf{MMV(C)}}(m)\}$ *is a retract of the algebra* $F_{\mathbf{MMV(C)}}(m)$. *In other words the m-generated monadic Boolean algebra* $(B(m), \exists)$ *is a projective algebra in* $\mathbf{MMV(C)}$.

Proof. Firstly we show that $(\mathbf{2}^k, \exists)$ is a retract of D_k. Recall that $(\mathbf{2}^k, \exists)$ is a homomorphic image by maximal monadic filter. Denote this homomorphism by $h : D_K \to (\mathbf{2}^k, \exists)$. Notice that the maximal monadic filter is generated by the set $\{x \in \exists D_k : 2x = 1\}$. On the other hand the Boolean envelope $(B(D_k), \exists)$, where $B(D_k) = \{2x^2 : x \in D_k\}$, is a subalgebra of D_k, which is isomorphic to $(\mathbf{2}^k, \exists)$. Denote by $\varepsilon : (B(D_k), \exists) \to D_k$ this embedding. It is obvious that $h\varepsilon = Id_{B(D_k)}$. □

Corollary 25. $(\mathbf{2}^{k_1}, \exists) \times ... \times (\mathbf{2}^{k_n}, \exists)$ *is a retract of* $D_{k_1} \times ... \times D_{k_n}$.

Proof. Let A_1, A_2 be any algebras and, respectively, B_1, B_2 are retracts of them, i. e. we have homomorphisms $h_i : A_i \to B_i$ and $\varepsilon_i : B_i \to A_i$ such that $h_i\varepsilon_i = Id_{B_i}$ $(i = 1, 2)$. Then $B_1 \times B_2$ is a retract of $A_1 \times A_2$. Indeed, $h = (h_1, h_2 n) : A_1 \times A_2 \to B_1 \times B_2$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ are homomorphisms such that $h\varepsilon = Id_{B_1 \times B_2}$. From here follows this Corollary.

□

Lemma 26. *For any* $k \in \{1, ..., 2^{2^m} - 1\}$ *there exists principal monadic filter* $[u)$ *of m-generated free* $MMV(C)$*-algebra* $F_{\boldsymbol{MMV(C)}}(m)$ $(= \prod_{k=1}^{2^{2^m}-1} D_k)$ *such that* $\pi_k(F_{\boldsymbol{MMV(C)}}(m)) \cong F_{\boldsymbol{MMV(C)}}(m)/[u)$, *where* $\pi_k : F_{\boldsymbol{MMV(C)}}(m) \to D_k$ *is a projection on k-th component* D_k *and* $u \in F_{\boldsymbol{MMV(C)}}(m)$.

Proof. Let $u = (0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^{2^m}-1}) \in F_{\mathbf{MMV(C)}}(m)$, where 1^k is the top element of D_k, 0^i is the bottom element of D_i. Notice that $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^{2^m}-1})$ is Boolean element that belongs to $F_{\mathbf{MMV(C)}}(m)$. Then $[u)$ will be a monadic filter such that $F_{\mathbf{MMV(C)}}(m)/[u) \cong D_k$. With this one we have proven this Lemma. □

Lemma 27. *The algebra* $D_1 \times D_{k_1} \times ... \times D_{k_n}$ *is a projective* $MMV(C)$*-algebra, where* $1 < k_i \leq 2^{2^m} - 1$, $1 \leq i \leq n$ *and* D_1 *is m-generated subdirectly irreducible perfect* $MMV(C)$*-algebra.*

Proof. Let $\pi_{1k_1...k_n} : F_{\mathbf{MMV(C)}}(m) \to D_1 \times D_{k_1} \times ... \times D_{k_n}$ be a projection onto $D_1 \times D_{k_1} \times ... \times D_{k_n}$. Let $\{r_1, ..., r_p\} = \{1, ..., 2^{2^m} - 1\} - \{1, k_1, ..., k_n\}$. So, $F_{\mathbf{MMV(C)}}(m) = D_1 \times \prod_{i=1}^{n} D_{k_i} \times \prod_{i=1}^{p} D_{r_i}$. Then $D_1 \times \prod_{i=1}^{n} D_{k_i} \times (\mathbf{2}, \exists)$ is a subalgebra of $D_1 \times \prod_{i=1}^{n} D_{k_i} \times \prod_{i=1}^{p} D_{r_i}$. Observe that $(D, \exists)$, where $D = \{(x, 1) : x \in \neg RadD_1\} \cup \{(x, 0) : x \in RadD_1\}$, is a subalgebra of $D_1 \times (\mathbf{2}, \exists)$, which is isomorphic to D_1. So, $D_1 \times \prod_{i=1}^{n} D_{k_i}$ is a subalgebra of $D_1 \times \prod_{i=1}^{n} D_{k_i} \times (\mathbf{2}, \exists)$. Then there exists the embedding $\varepsilon : D_1 \times D_{k_1} \times ... \times D_{k_n} \to D_1 \times \prod_{i=1}^{n} D_{k_i} \times \prod_{i=1}^{p} D_{r_i}$. Now, it is easy to check that $\pi_{1k_1...k_n}\varepsilon = Id_{D_1 \times D_{k_1} \times ... \times D_{k_n}}$. Hereby it is proved the theorem. □

As in the variety $\mathbf{MV(C)}$ of $MV(C)$-algebras we have

Theorem 28. *m-generated subalgebra $(A, \exists)$ of $F_{\mathbf{MMV(C)}}(m)$ is projective iff $(A, \exists)$ is finitely presented and $A \cong A_0 \times A_1$ where A_0 is a perfect MV-algebra.*

Proof. First of all notice that if A is not represented as $A_0 \times A_1$, where A_0 is a perfect MV-algebra, then A can not be a subalgebra of $F_{\mathbf{MMV(C)}}(m)$ and thereby it will not be a retract of $F_{\mathbf{MMV(C)}}(m)$. Indeed, let A_0 be a retract of $F_{\mathbf{MMV(C)}}(m)$, i. e. there exist homomorphisms $h_1 : F_{\mathbf{MMV(C)}}(m) \to A_0$ and $\varepsilon_1 : A_0 \to F_{\mathbf{MMV(C)}}(m)$ such that $h_1\varepsilon_1 = Id_{A_0}$. Since the variety $\mathbf{MB}$ of monadic Boolean algebras is a subvariety of $\mathbf{MMV(C)}$, we have that there exists a homomorphism $f : F_{\mathbf{MMV(C)}}(m) \to F_{\mathbf{MB}}(m)$. Let $B(A_0) = f\varepsilon_1(A_0)$. Denote the composition $f\varepsilon_1$ by k. So, for homomorphisms $f : F_{\mathbf{MMV(C)}}(m) \to F_{\mathbf{MB}}(m)$ and $kh_1 : F_{\mathbf{MMV(C)}}(m) \to B(A_0)$ there exists homomorphism $h_2 : F_{\mathbf{MB}}(m) \to B(A_0)$ such that $h_2 f = kh_1$. For $f\varepsilon_1 : A_0 \to F_{\mathbf{MMV(C)}}(m)$ and $k : A_0 \to B(A)$ there exists a homomorphism $\varepsilon_2 : B(A_0) \to F_{\mathbf{MB}}(m)$ such that $f\varepsilon_1 = \varepsilon_2 k$. From $h_2 f = kh_1$ we have $h_2 f\varepsilon_1 = kh_1\varepsilon_1$, and hence $h_2 f\varepsilon_1 = k$, since $h_1\varepsilon_1 = Id_{A_0}$. Then $h_2\varepsilon_2 k = k$, because $f\varepsilon_1 = \varepsilon_2 k$. Since k is a surjective homomorphism, we have that $h_2\varepsilon_2 = Id_{B(A_0)}$. So, $B(A_0)$ is a retract of $F_{\mathbf{MB}}(m)$ and, hence, it is projective. According to Proposition 20 m-generated monadic Boolean algebra $(B, \exists)$ is projective in the variety of monadic Boolean algebras iff $(B, \exists) \cong (\mathbf{2}, \exists) \times (B', \exists)$ for some m-generated monadic Boolean algebra $(B', \exists)$. But $(\mathbf{2}, \exists)$ is a homomorphic image of perfect monadic $MV(C)$-algebra. Notice also that any m-generated projective $MMV(C)$-algebra is finitely presented.

Now let us suppose that $(A, \exists)$ is finitely presented and $A \cong A_0 \times A_1$ where A_0 is a perfect MV-algebra. Then $(A, \exists)$ is a homomorphic image of $F_{\mathbf{MMV(C)}}(m)$ by some principal monadic filter $[u)$ for some $u \in F_{\mathbf{MMV(C)}}(m)$.

According to Theorem 15 free algebra $F_{\mathbf{MMV(C)}}(m)$ is isomorphic to the finite product of monadic $MV(C)$-algebras D_k $(1 \leq k \leq 2^{2^m} - 1)$ the homomorphic image by maximal monadic filter of which is isomorphic to the subdirectly irreducible monadic Boolean algebra $(\mathbf{2}^k, \exists)$. Then $(A, \exists)$ is a homomorphic image of $D_1 \times D_{k_1} \times ... \times D_{k_n}$ which is projective (Lemma 25), where D_1 is a perfect $MMV(C)$-algebra. So, there exists principal monadic filter $[u')$ of $F_{\mathbf{MMV(C)}}(m)$ such that $F_{\mathbf{MMV(C)}}(m)/[u') \cong D_1 \times D_{k_1} \times ... \times D_{k_n}$. Then there exists principal monadic filter $[u_A) = [\pi_{1k_1...k_n}(u'))$ of the algebra $D_1 \times D_{k_1} \times ... \times D_{k_n}$ such that $D_1 \times D_{k_1} \times ... \times D_{k_n}/[\pi_{1k_1...k_n}(u')) \cong A$, where $\pi_{1k_1...k_n} : F_{\mathbf{MMV(C)}}(m) \to D_1 \times D_{k_1} \times ... \times D_{k_n}$ is a projection of $F_{\mathbf{MMV(C)}}(m)$ onto $D_1 \times D_{k_1} \times ... \times D_{k_n}$. Let $u_1 = \pi_1(u_A),\ u_{k_i} = \pi_{k_i}(u_A)$ be projections of the element u_A on corresponding components $D_1, D_{k_1}, ..., D_{k_n}$ respectively. Then $D_1/[u_1),\ D_{k_i}/[u_{k_i})$ are retracts of $D_1,\ D_{k_i}$ $(i = 1, ..., n)$ respectively (Lemma 20). Then $D_1/[u_1) \times \prod_{i=1}^{n} D_{k_i}/[u_{k_i})$ is a retract of $D_1 \times \prod_{i=1}^{n} D_{k_i}$. Therefore A is projective (Lemma 19, 20, 25).

□

6 Projective formulas

Let us denote by $\mathcal{P}_m$ a fixed set $x_1, ..., x_m$ of propositional variables and by Φ_m the set of all propositional formulas in L_P with variables in $\mathcal{P}_m$. Notice that the m-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(m)$ is isomorphic to $\Phi_m/\equiv$, where $\alpha \equiv \beta$ iff $\vdash (\alpha \leftrightarrow \beta)$ and $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$. Subsequently we do not distinguish between the formulas and their equivalence classes. Hence we simply write Φ_m for $F_{\mathbf{MV(C)}}(m)$, and $\mathcal{P}_m$ plays the role of the set of free generators. Since Φ_m is a lattice, we have an order $\leq$ on Φ_m . It follows from the definition of $\to$ that for all $\alpha, \beta \in \Phi_m$, $\alpha \leq \beta$ iff $\vdash (\alpha \to \beta)$.

Let α be a formula of the logic L_P and consider a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ and extend it to all of Φ_m by $\sigma(\alpha(x_1, ..., x_m)) = \alpha(\sigma(x_1), ..., \sigma(x_m))$. We can consider the substitution as an endomorphism $\sigma : \Phi_m \to \Phi_m$ of the free algebra Φ_m.

Definition 29. *A formula $\alpha \in \Phi_m$ is called* projective *if there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$.*

Notice that the notion of projective formula was introduced for intuitionistic logic in [11].

Observe that we can rewrite any equation $P(x_1, ..., x_m) = Q(x_1, ..., x_m)$ in the variety **MV(C)** into an equivalent one $P(x_1, ..., x_m) \leftrightarrow Q(x_1, ..., x_m) = 1$. So, for **MV(C)** we can replace n equations by one

$$\bigwedge_{i=1}^{n} P_i(x_1, ..., x_m) \leftrightarrow Q_i(x_1, ..., x_m) = 1.$$

Now we are ready to show a close connection between projective formulas and projective subalgebras of the free algebra Φ_m.

Theorem 30. *Let A be an m-generated projective subalgebra of the free algebra Φ_m. Then there exists a projective formula α of m variables, such that A is isomorphic to $\Phi_m/[\alpha)$, where $[\alpha)$ is the principal filter generated by $\alpha \in \Phi_m$.*

Proof. Suppose A is an m-generated projective subalgebra of Φ_m with generators $a_1, ..., a_m$. Then A is a retract of Φ_m, and there exist homomorphisms $\varepsilon : A \to \Phi_m$, $h : \Phi_m \to A$ such that $h\varepsilon = Id_A$, where $\varepsilon(x) = x$ for every $x \in A \subset \Phi_m$. Observe that εh is an endomorphism of Φ_m. We will show now that $\alpha = \bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))$ is a projective formula, namely, that $\vdash \varepsilon h(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$, for all $\beta \in \Phi_m$.

Indeed, $\varepsilon h(\bigwedge_{j=1}^{m}(p_j \leftrightarrow \varepsilon h(p_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h \varepsilon h(x_j))$, and since $h\varepsilon = Id_A$, we have $\varepsilon h(\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h(x_j))$. Thus $\vdash \varepsilon h(\alpha)$. Further, for any $\beta \in \Phi_m$, $\varepsilon h(\beta(x_1, ..., x_m)) = \beta(\varepsilon h(x_1), ..., \varepsilon h(x_m))$, and since $\alpha \vdash x_j \leftrightarrow \varepsilon h(x_j)$, $j = 1, ..., m$, we have $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$.

Since A is an m-generated projective $MV(C)$-algebra, according to the Proposition 16, there exist m polynomials $P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)$ such that

$$P_i(x_1, ..., x_m) = \varepsilon(a_i) = \varepsilon h(x_i)$$

and

$$P_i(P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)) = P_i(x_1, ..., x_m),\ i = 1, ..., m.$$

Observe, that $h(x_i) = a_i$. Since the m-generated projective MV-algebra A is finitely presented by the equation $\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j)) = 1$, we have that $A \cong \Phi_m/[\alpha)$. $\square$

Theorem 31. *If α is a projective formula of m variables, then $\Phi_m/[\alpha)$ is a projective algebra which is isomorphic to a projective subalgebra of Φ_m.*

Proof. Suppose that α is a projective formula of m variables. Then there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$. Since σ is an endomorphism of Φ_m, $\sigma(\Phi_m)$ is a subalgebra of Φ_m. Now we will show that $\sigma(\Phi_m)$ is a retract of Φ_m, i. e. $\sigma^2 = \sigma$. Indeed, since α is a projective formula, $\sigma(\alpha) = 1_{\Phi_m}$, and $\alpha \leq \beta \leftrightarrow \sigma(\beta)$ for all $\beta \in \Phi_m$. But then $\sigma(\alpha) \leq \sigma(\beta) \leftrightarrow \sigma^2(\beta)$, $\sigma(\beta) \leftrightarrow \sigma^2(\beta) = 1_{\Phi_m}$, $\sigma(\beta) = \sigma^2(\beta)$, and $\sigma^2 = \sigma$. Hence $\sigma(\Phi_m)$ is a retract of Φ_m. So, $\sigma(\Phi_m)$ is isomorphic to $\Phi_m/[\alpha)$. □

Thus we have the following correspondence between projective formulas and projective subalgebras of Φ_m. To each m-generated projective subalgebra of m-generated free $MV(C)$-algebra corresponds an m-variable projective formula and to two non-isomorphic m-generated projective subalgebra of m-generated free $MV(C)$-algebra correspond non-equivalent m-variable projective formulas. And two non-equivalent m-variable projective formulas correspond two different m-generated projective subalgebra of m-generated free $MV(C)$-algebra (but they can be isomorphic).

Therefore we arrive at the following

Corollary 32. *There exists a one-to-one correspondence between projective formulas with m variables and m-generated projective subalgebras of Φ_m.*

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