HILBERT FUNCTIONS OF MORAVA $K(2)^*$-THEORY RINGS OF SOME 2-GROUPS

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Abstract. This note presents some calculation of the Hilbert series related to Morava $K(2)^*$-theory rings of classifying spaces of some 2-groups.

1. Introduction

This is a joint project with M. Bakuradze in preparation. The author takes this opportunity as a participant of Fourth Annual Conference in Exact and Natural Sciences Dedicated to 140th Anniversary of the Birth of Ivane Javakhishvili to present some examples and calculations in more details.

A commutative ring R is called N-graded if it has a direct sum decomposition $R = \oplus_i R_i$ (that is, the R_i are additive subgroups, and every element r of R can be written in a unique way as finite sum $r = r_1 + \cdots + r_m$, where the r_j are nonzero and belong to distinct R_i) and moreover $R_i R_j \subseteq R_{i+j}$. Elements that belong to one of the R_i are called homogeneous. The r_j that occur in the unique representation of r are called the homogeneous components of r.

Consider the situation of a N-graded k-algebra R: all R_i are vector spaces over a field k. For N graded R module M put $H(M,t) = \sum_i H(M,i)t^i$ where $H(M,i) = dim_k M_i$. The function $H(M,.) : N \to N$ is called the Hilbert function of M. Here are some examples

$R = k$	$H(R,t) = 1.$
$R = k[x]$	$H(R,t) = 1 + t + t^2 + \cdots = 1/(1-t).$
$R = k[x,y]$	$H(R,t) = 1 + 2t + 3t^2 + 4t^3 + \cdots = 1/(1-t)^2.$
$R = k[x_1, \cdots, x_m]$	$H(R,t) = 1/(1-t)^m.$
$R = k[x,y]/(xy)$	$H(R,t) = 1 + 2t + 2t^2 + 2t^3 + \cdots = (1+t)/(1-t).$
$R = k[x,y]/(x^2+y^2)$	$H(R,t) = 1 + 2t + 2t^2 + 2t^3 + \cdots = (1+t)/(1-t).$

The last two examples show that non-isomorphic graded k-algebras can have the same Hilbert series.

In our situation of $K^*(2)(BG)$ the naive definition of $HP(t)$ does not work as there are the elements of negative degree in the ring of coefficients $K(2)^*$. That is the relations ideal I is not homogeneous with respect to variables $x_1, \cdots x_n$, hence

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the quotient ring is not graded (but filtered). So we have to reduce the definition to graded algebra case. In such situations the following definition is used [8]. Let I be an ideal of a polynomial ring $k[x_1, \cdots, x_n]$ over a field k, and let $>$ be a global monomial ordering. The ring $k[x_1, \cdots, x_n]/I$ is a filtered algebra, however its Hilbert-Poincare series w.r.t $>$ is defined as follows. Replace I by its leading ideal $L(I) := L_>(I) = (L_>(f) | f \in I) \subset k[x_1, \cdots, x_n]$, that is $L(I)$ is generated by leading terms of the elements of I. Then the ring $k[x_1, \cdots, x_n]/L_>(I)$ is a graded ring. By definition $HP(t, k[x_1, \cdots, x_n]/I) = HP(t, k[x_1, \cdots, x_n]/L_>(I))$ w.r.t. $>$.

In this case Hilbert series are actually the polynomials because $K(s)^*(BG)$ are finite dimensional $K(s)^*$-vector space for finite group G.

2. Our examples

In [7] all groups of order 32 are listed and numbered by $1, \cdots, 51$. For the groups $G_{34}, \cdots, G_{41}$, the dihedral group, the quasi-dihedral group, the semi-dihedral group, the generalized quaternion group, the rings $K(s)^*(BG)$ are calculated in [1], [5], [4], [6] and [3]. Because of complexity of these rings we need a simple but still informative numerical invariants such as Hilbert-Poincare series.

We consider in details the groups G_{34} and its non-split version G_{35}. The other examples can be treated similarly. Recall the following presentations

$$G_{34} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle,$$
$$G_{35} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{c}^2 = \mathbf{a}^2, \mathbf{cac}^{-1} = \mathbf{a}^{-1}, \mathbf{cbc}^{-1} = \mathbf{b}^{-1} \rangle.$$

Proposition 2.1. *One has* $K(2)^*(BG_{34}) \cong K(2)^*[a, b, c, x_1, y_1, x_2, y_2, T]/I$, *where* $|a| = |b| = |c| = |x_1| = |y_1| = 1$, $|x_2| = |y_2| = |T| = 2$ *and the relations ideal* I *is as follows*

$I = (a^4, b^4, c^4, c + x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4, y_1 + c + v{y_2}^2 + v^3{y_1}^2{y_2}^4, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + b^2 + bc, v^2{x_2}^4 + a^2 + ac, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

Here our base field is the graded field $K(2)^*(pt) = F_2[v.v^{-1}]$, $|v| = -3$. Our filtered ring $K(2)(BG_{34})$ is the vector space over $K^*(2)(pt)$ of dimension 184. This can be checked by SINGULAR [8] code

$$> ring\ \ R = (2, v), (list\ \ of\ \ variables), (a(-3, list\ \ of\ \ weights), dp);$$
$$> ideal\ \ I = (list\ \ of\ \ ideal\ \ generators);$$
$$> vdim(std(I));$$
$$> 184$$

Proposition 2.2. *Hilbert polynomial H(t) of* $K(2)^*(BG)$*, for* $G = G_{34}, G_{35}$ *w.r.t.* $(a, b, c, y_1, x_1, y_2, x_2, T), (1, 1, 1, 1, 1, 2, 2, 2), dp$ *is given by*

$$H(t) = t^{12}+4t^{11}+7t^{10}+12t^9+14t^8+17t^7+23t^6+27t^5+32t^4+25t^3+16t^2+5t+1.$$

Proof. In principle the Hilbert function, that is, the number of n-weighted basis elements can be read off from Proposition 2.1. On the other hand this can be easily checked in two ways. First way is to calculate Hilbert function $H(n)$ by SINGULAR code

$$> ring\ \ R = (2, v), (variables), (a(-3, weights), dp);$$
$$> weightedKB(std(I), n, intvec(weights), intvec(-3));$$

Alternatively one can first compute the so called first Hilbert series $Q(t)$ by code $> hilb(std(I), 1, intvec(1, 1, 1, 1, 1, 2, 2, 2,))$ and then use the celebrated formula

$$H(t) = \frac{Q(t)}{(1 - t^{w_i})},$$

relating Hilbert first series with Hilbert-Poincare series (Hilbert second series). Here w_i are weights of the variables. In our case the variables in Proposition 2.1 have weights $(1, 1, 1, 1, 1, 2, 2, 2)$, therefore one has in our case

$$H(t) = \frac{Q(t)}{(1 - t)^5(1 - t^2)^3}.$$

Then SINGULAR returns

$$\begin{aligned}Q(t) =& 1 - 2t^2 - 15t^3 + 22t^4 + 26t^5 - 31t^6 - 27t^7 - 10t^8 + 60t^9 + 10t^{10} - 42t^{11}\\ &+ 8t^{12} - 28t^{13} + 32t^{14} + 18t^{15} - 31t^{16} + 12t^{17} - 8t^{18} + t^{19} + 10t^{20}\\ &- 6t^{21} - t^{22} + t^{23}.\end{aligned}$$

Factorizing $Q(t)$ one has

$$\begin{aligned}Q(t) =&(t - 1)^8(t + 1)^3(t^{12} + 4t^{11} + 7t^{10} + 12t9 + 14t^8 + 17t^7 + 23t^6 + 27t^5\\ &+ 32t^4 + 25t^3 + 16t^2 + 5t).\end{aligned}$$

Thus we obtain $H(t)$ as claimed for G_{34}.
For G_{35}, the non-split version of G_{34}, we have the following relations ideal

$$\begin{aligned} I =&(a^4, b^4, c^4, c + x_1 + vx_2^2 + v^2x_1^2x_2^4, y_1 + vy_2^2 + v^3y_1^2y_2^4,\\ &c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2),\\ &v^2y_2^4 + b^2 + bc + c^2, v^2x_2^4 + a^2 + ac, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T,\\ &(c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T,\\ &T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2),\\ &T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT). \end{aligned}$$

Here the variables have the same weights as in Proposition 2.1. The ring structure is not isomorphic to what we had in the case of G_{34}. However $K(2)^*(BG_{35})$ has the same Hilbert-Poincare series. This is not so surprising: the Serre spectral sequence for $K(2)^*(BG)$ does not see the difference between the group G_{35} and its non-split version of G_{34}.

□

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