The hydrogen atom in spheroidal coordinates: the continuous spectrum

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An exceptional role of the hydrogen atom becomes evident according to its formation directly after the *big bang*. Hydrogen accounts for nine of ten atoms in the present universe and for nearly three-quarters of its mass. The solution of the hydrogen atom problem is hence of such fundamental importance that it must be solved in all possible cases. In textbooks of quantum mechanics the problem is typically treated in spherical polar and paraboloidal coordinates for the discrete and continuous spectrum.

In addition to spherical polar and paraboloidal coordinates, the Schrödinger equation for a Coulomb field is separable in prolate spheroidal coordinates. A Coulomb centre is located at one focus of those spheroidal coordinates; another focus lies at distance R from the Coulomb centre. The separation yields three equations for the three spatial variables, which become the familiar radial and angular equations in spherical polar coordinates when R tends to zero; their solutions might hence be called quasi-radial and quasi-angular functions.

Coulson and Robinson [1] made an initial attempt to solve the hydrogen atom problem in spheroidal coordinates for the discrete spectrum; their important result was a termination of the power series for the one-dimensional functions and a derivation of Coulomb spheroidal functions in a polynomial form. In [2] we show that a simple and straightforward scheme of calculation enables spheroidal functions to be derived in a polynomial form for, in principle, arbitrary eigenstates. Moreover, we obtained Coulomb Sturmian functions in prolate spheroidal coordinates in a closed algebraic form, and investigated their properties and application [3,4].

The hydrogen atom problem in spheroidal coordinates appears not to have been treated for the continuous spectrum; Coulomb spheroidal functions have not yet been obtained in a closed algebraic form. This work is devoted to eliminate this deficiency in our knowledge.

For the first time Coulomb spheroidal functions are obtained on exact solution of the appropriate one-dimensional equations, which are shown to be Heun's confluent equations. The one-dimensional amplitude functions are presented as a convenient power series, which leads to a pair of three-term recurrence relations for the quasi-angular and quasi-radial functions. The separation parameter is determined from a continued-fraction equation, the roots of which yield these eigenvalues. The derived functions are a natural generalization of the well-known Coulomb wave functions of the continuous spectrum obtained in spherical polar coordinates. An application of Coulomb spheroidal functions for the analytical solution of the more general two-Coulomb-centre problem for the continuous spectrum is discussed.

References

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