Constructive stochastic integral representation of Wiener functional

O. Purtukhia, V. Jaoshvili

Ivane Javakhishvili Tbilisi State University, A.Razmadze Mathematical Institute;
Faculty of Exact and Natural Sciences, Department of
Mathematics, 3 Chavtchavadze Ave, Tbilisi 0179, Georgia

As it is well-known, one important property of Ito stochastic integrals is the fact that they form martingales. In particular, if H be the class of all functions $f(t,\omega):[0,\infty)\times\Omega\to R$ such that: (*i*) the mapping $(t,\omega)\mapsto f(t,\omega)$ is progressively measurable; (*ii*) the random variable $f(t,\cdot)$ is $\Im_t^w$-measurable (where $\{w_t, t\geq 0\}$ is a standard Wiener process and $\Im_t^w$ is a natural filtration generated by him, $\Im_t^w=\overline{\sigma}\{w_s, s\in[0,t]\}$); (*iii*) the mapping f is square integrable with respect to $P\otimes\lambda$ (that is $E[\int_0^T f^2(t,\omega)\lambda(dt)]<\infty$), then the stochastic process $\{\int_0^t f(s,\omega)dw_s(\omega), t\geq 0\}$ is an $\Im_t^w$-martingale. On the other hand, according to the well-known Clark formula (1971), the inverse statement (so-called martingale representation theorem) is also right. Indeed, if F is a square integrable $\Im_T^w$-measurable random variable, then (due to the Clark formula) there exist a square integrable $\Im_t^w$-adapted random process $\varphi(t,\omega)$ such that $F=EF+\int_0^T\varphi(s,\omega)dw_s$. Taking the conditional mathematical expectation from the both sides of the last relation we obtain that for the associated to F Levy's martingale $M_t=E(F\,|\,\Im_t^w)$ the following stochastic integral representation is true $M_t=M_0+\int_0^t\varphi(s,\omega)dw_s$.

It should be noted that the Clark formula (as well as the martingale representation theorem) yields a fairly abstract result. A closed form expression for the integrand can only be obtained in special cases of which we discuss below. Most of the general research on constructive martingale representation has been within Malliavin calculus. Here the constructive martingale representation is based on the Malliavin derivative and is known as the Clark-Haussmann-Ocone formula (see, Ocone, 1984 in Wiener case and Ma, Protter and Martin, 1998 in the case of normal martingales for functionals from the class $D_{2,1}^M$). We (Purtukhia, 2003) have introduced the space $D_{p,1}^M$ $,1<p<2$ and extended the Clark-Haussmann-Ocone formula for functionals from this space. Absolutely different method for finding of $\varphi(t,\omega)$ was offered by Shyriaev, Yor and Graversen (2003, 2006), which was based on using of Ito's (generalized) formula and Levy's theorem for associated to F Levy's martingale. We (Purtukhia, Jaoshvili, 2009) introduced the new construction of stochastic derivative of Poisson functional and established the explicit expression for the integrand of Clark representation.

The class of martingales to which the Clark-Haussmann-Ocone formula can be applied is, however, limited by the condition that the terminal value of the martingale must be Malliavin differentiable. We (Glonti and Purtukhia, 2014) considered case when terminal value M_T is not Malliavin differentiable, but from its conditional mathematical expectation $E(M_T\,|\,\Im_t^w)$ one can to select a Malliavin differentiable subsequence (with respect to $t\in[0,T)$) and generalized the Clark-Haussmann-Ocone formula. Here we consider the one path-dependent Wiener functional $F=(w_T-C_1)^-\cdot I_{\{\inf_{0\leq t\leq T} w_t\leq C_2\}}$ which is not Malliavin differentiable and establish for it the stochastic integral representation formula with explicit form of integrand. For achievement of this purpose, we calculate the conditional mathematical expectation of the considered functional and apply above-mentioned our generalization of the Clark-Haussmann-Ocone formula.